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REPRESENTATION OF LATTICES ON PRE A*-ALGEBRA

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Abstract: This paper analyzes the notion of lattice structure on Pre A*-algebra. It has been derived the corresponding properties of the Pre A*-lattice L. Furthermore, identified a congruence relation β_a on L and proved that the set of all congruences on L is a distributive Pre A*-Lattice. Also described an ideal on Pre A*-lattice L and shown that $F(L)$ the set of all ideals of L is a distributive Pre A*-lattice under the set inclusion. Also introduced the notion of ideal congruence on Pre A*-lattice and derived its various significant properties.

Keywords: A*-Algebra, Pre-A*-Algebra, Boolean Algebra, Partially Ordered Set, Homomorphism.

AMS Subject Classification (2000): 06E05, 06E25, 06E99, 06B10.

Introduction: J.Venkateswara Rao (2000) introduced the concept Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$ analogous to C-algebra as a reduct of A*- algebra. Further A. Satyanarayana (2012) established the concept of Ideals, Semilattice structures and Ideal congruences on Pre A*-algebra. Boolean algebra depends on two element logic. C-algebra, Ada, A*- algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth values, where the third truth value stands for an undefined truth value. The Pre A*- algebra structure is denoted by $(A, \wedge, \vee, (-)^{\sim})$ where A is non-empty set $\wedge, \vee$, are binary operations and $(-)^{\sim}$ is a unary operation.

In this paper we identify for any subset L of a Pre A*-algebra, a Pre A*-lattice. We present various examples of Pre A*-lattices. We offer several properties of Pre A*-lattices. We define sub Pre A*-lattice, distributive Pre A*-lattices and homomorphism of Pre A*-lattices. We confer congruence relation β_a on L and prove that the set of all congurences of the form β_a forms a distributive Pre A*-Lattice . We also introduce the concept of Ideal, Ideal congruences on Pre A*-lattice and derived some important properties of these.

1. **Preliminaries:** In this section we concentrate on the algebraic structure of Pre A*-algebra and state some results which will be used in the later text.

1.1. **Definition:** An algebra $(A, \wedge, \vee, (-)^{\sim})$ where A is a non-empty set with $1, \wedge, \vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim} = x \quad \forall x \in A$
(b) $x \wedge x = x, \quad \forall x \in A$
(c) $x \wedge y = y \wedge x, \quad \forall x, y \in A$
(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim} \quad \forall x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \quad \forall x, y \in A$ is called a Pre A*-algebra.

1.2. Example: $3 = \{0, 1, 2\}$ with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

1.3. Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$

(b) $1 \wedge x = x$ for all $x \in 3$

(c) $0 \vee x = x$ for all $x \in 3$

(d) $2 \wedge x = 2 \vee x = 2$ for all $x \in 3$.

1.4. Definition: Let A be a Pre A*-algebra. An element $x \in A$ is called a central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by B (A).

1.5. Theorem: [6]: Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

2. *Representation of Lattices On Pre A*-Algebra*: In this section we define (for any subset L of a Pre A*-algebra) a Pre A*-lattice (L, ∧, ∨). We give some examples of Pre A*-lattices. We give some properties of Pre A*-lattices. We define the congruence $\beta_a = \{(x, y) \in L \times L \;/\; a \vee x = a \vee y\}$ for any $a \in L$ and studied their properties. We also introduce the concept of Ideal, Ideal congruences on Pre A*-lattice and derived some important properties of these.

2.1. Definition: Let A be a Pre A*-algebra. A non-empty subset L of a Pre A*-algebra A in which for each pair of elements a ∈ A, b ∈ B(A) in L has greatest lower bound a ∨ b exists in L. Such a defined set L in Pre A*-algebra is said to be Pre A*-lattice.

Now we give another type of definition other than that in 2.1. Definition by means of equations.

2.2. Definition: Let A be a Pre A*-algebra. A non-empty subset L of a Pre A*-algebra A, equipped with two binary operations meet (∧) and join (∨) which assign to every pair a ∈A, b ∈B(A) of the elements of L, uniquely an element a ∧ b as well as element a ∨ b in L in such a way that the following axioms holds.

(i) a ∧ (b ∧ c) = (a ∧ b) ∧ c, ∀ a, b, c ∈L (associative)

(ii) a ∧ b = b ∧ a, ∀ a, b ∈ L (commutative);

(iii) a ∧ (a ∨ b) = a, ∀ a, b ∈ L (absorption law)

2.3. Theorem: Let A be a Pre A*-algebra. L is a subset of A Then (L, ∧, ∨) is a Pre A*-lattice.

Proof: Since A is a Pre A*-algebra L is a subset of A.

We have a ∧ (b ∧ c) = (a ∧ b) ∧ c, ∀ a, b, c ∈ A by 1.2 (e)

a ∧ b = b ∧ a, ∀ a, b ∈ A by 1.2 (c) and a ∧ a = a, ∀ a ∈ A by 1.2 (b)

Therefore a ∧ (b ∧ c) = (a ∧ b) ∧ c, ∀ a, b, c ∈ L, a ∧ b = b ∧ a, ∀ a, b ∈ L

And a ∧ a = a, ∀ a ∈ L. Hence (L, ∧) is a semilattice.

Similarly we can prove (L, ∨) is a semilattice. And since in a Pre A*-algebra, a ∧ (a ∨ b) = a, ∀ a ∈ A, b ∈ B(A). So, a ∧ (a ∨ b) = a, ∀ a, b ∈ L (absorption law). Hence (L, ∧, ∨) is a Pre A*-lattice.

2.4. Definition: Let A be a Pre A*-algebra and L is a distributive Pre A*-Lattice. Define for any $a \in L$, $\beta_a = \{(x, y) \in L \times L \,/\, a \vee x = a \vee y\}$

2.5. Lemma: Let L be a distributive Pre A*-Lattice. Then $\beta_a = \{(x, y) \in L \times L \,/\, a \vee x = a \vee y\}$ is a congruence relation on L.

2.6. Theorem: Let L be a distributive Pre A*-Lattice. Define $X = \{\beta_a \,/\, a \in L\}$ then $(X, \subseteq)$ is a distributive Pre A*-Lattice.

Proof: Clearly $\beta_a \subseteq \beta_a$ for all $\beta_a \in X$, shows $\subseteq$ is reflexive.

Let $\beta_a \subseteq \beta_b$ and $\beta_b \subseteq \beta_a$ implies that $\beta_a = \beta_b$ .This $\subseteq$ is anti- symmetric.

Let $\beta_a \subseteq \beta_b$ and $\beta_b \subseteq \beta_c$ then $\beta_a \subseteq \beta_c$, shows that $\subseteq$ is transitive.

Hence $(X, \subseteq)$ is a poset.

First we show that $\beta_a \cap \beta_b = \beta_{a\ b}$

Let $(x, y) \in \beta_a \cap \beta_b$ then $a \vee x = a \vee y$ and $b \vee x = b \vee y$

Now $(a \wedge b) \vee x = (a \vee x) \wedge (b \vee x) = (a \vee y) \wedge (b \vee y) = (a \wedge b) \vee y$

This implies that $(x, y) \in \beta_{a\ b}$ This shows that $\beta_a \cap \beta_b \subseteq \beta_{a\ b}$.

On the other hand let $(x, y) \in \beta_{a\ b}$ then $(a \wedge b) \vee x = (a \wedge b) \vee y$.

Now $a \vee x = (a \vee (a \wedge b)) \vee x$ (by absorption law)

$$
\begin{aligned}
&= a \vee ((a \wedge b) \vee x) \\
&= a \vee ((a \wedge b) \vee y) \\
&= (a \vee (a \wedge b)) \vee y \\
&= a \vee y
\end{aligned}
$$

This implies that $(x, y) \in \beta_a$, hence $\beta_{a\ b} \subseteq \beta_a$

Similarly we can prove that $\beta_{a\ b} \subseteq \beta_b$ This implies that $\beta_{a\ b} \subseteq \beta_a \cap \beta_b$

Hence $\beta_a \cap \beta_b = \beta_{a\ b}$

Let $(r, s) \in \beta_a$ then $a \vee r = a \vee s \Rightarrow a \vee b \vee r = a \vee b \vee s \Rightarrow (r, s) \in \beta_{a\ b}$.

Hence $\beta_a \subseteq \beta_{a\ b}$. Similarly $\beta_b \subseteq \beta_{a\ b}$ .Thus $\beta_{a\ b}$ is an upper bound of $\{\beta_a, \beta_b\}$.

Let β_c is an upper bound of $\{\beta_a, \beta_b\}$.

Let $(x, y) \in \beta_{a\ b}$ then $(a \vee b) \vee x = (a \vee b) \vee y$

$\Rightarrow a \vee (b \vee x) = a \vee (b \vee y) \Rightarrow (b \vee x, b \vee y) \in \beta_a \subseteq \beta_c$

$\Rightarrow c \vee b \vee x = c \vee b \vee y \Rightarrow b \vee c \vee x = b \vee c \vee y \Rightarrow (c \vee x, c \vee y) \in \beta_b \subseteq \beta_c$

$\Rightarrow c \vee c \vee x = c \vee c \vee y \Rightarrow c \vee x = c \vee y \Rightarrow (x, y) \in \beta_c$

Hence $\beta_{a\ b} \subseteq \beta_c$.

Therefore Sup$\{\beta_a, \beta_b\} = \beta_{a\ b}$ i.e., $\beta_a \vee \beta_b = \beta_{a\ b}$.

Hence X is a Pre A*-Lattice.

Since $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $\forall$ a, b, c $\in$ L we have X is a distributive Pre A*-Lattice.

2.7. Definition: A nonempty subset I of a distributive Pre A*-Lattice L is said to be an ideal of L if the following hold.

(i) $a, b \in I \Rightarrow a \vee b \in I$

(ii) $a \in I \Rightarrow x \wedge a \in I$ for each $x \in L$

2.8. Theorem: Let L be a distributive Pre A*-Lattice. Then $F(L)$ the set of all ideals of L is a distributive Pre A*-Lattice under the set inclusion.

Proof: Let $I, J \in F(L)$

Clearly I I J is an ideal of L, and I I J = Inf{I, J} in the poset $(F(L), \subseteq)$.

Let $H = \{ \bigvee_{i=1}^{n} x_i \;/\; x_i \in I \cup J$, n is a positive integer $\}$

Let $x, y \in H$ implies that $x = \bigvee_{i=1}^{n} x_i$, $y = \bigvee_{i=1}^{n} y_i$ and hence $x \vee y = \bigvee_{k=1}^{n} t_k$ (each $t_i \in I \cup J$)

Let $a \in L$ and $x \in H$.Then $a \wedge x = a \wedge (\bigvee_{i=1}^{n} x_i) = \bigvee_{i=1}^{n} (a \wedge x_i)$

Now $a \wedge x_i \in I \cup J$ (since if $x_i \in I$ then $a \wedge x_i \in I$ and if $x_i \in J$ then $a \wedge x_i \in J$)

Hence $a \wedge x \in H$.Therefore H is an ideal of L.

Clearly I, J are subsets of H.Let K be any ideal of L such that $I \subseteq K$, $J \subseteq K$.

Now let $x \in H \Rightarrow x = \bigvee_{i=1}^{n} x_i \Rightarrow x \in K$(since $I, J \subseteq K$ and K is an ideal)

Hence $H \subseteq K$.Therefore H is the smallest ideal containing I, J.

Therefore Sup{I, J} = H.i.e., $I \vee J = \{ \bigvee_{i=1}^{n} x_i \;/\; x_i \in I \cup J$, n is a positive integer $\}$

Hence $F(L)$is a lattice under the set inclusion.

Let $I, J, K \in F(L)$.Clearly (I I J) $\vee$ (I I K) $\subseteq$ I I ($J \vee K$)

Let $t \in$ I I ($J \vee K$)

Then $t = \bigvee_{i=1}^{n} x_i$, where $x_i \in J \cup K$.Now $t = t \wedge t = t \wedge (\bigvee_{i=1}^{n} x_i) = \bigvee_{i=1}^{n} (t \wedge x_i)$

Now $t \wedge x_i \in$ I I J or I I K

Therefore $t \in$ (I I J) $\vee$ (I I K)

Hence I I ($J \vee K$) $\subseteq$(I I J) $\vee$ (I I K)

Thus I I ($J \vee K$) = (I I J) $\vee$ (I I K).Therefore $F(L)$ is a distributive Pre A*-Lattice.

2.9. Definition: For any ideal I of a Pre A*-algebra A we define $\beta_I = \{(x, y) \;/$ $a \vee x = a \vee y$ for some $a \in I$ }.That is $\beta_I = \bigcup_{a \in I} \beta_a$

2.10. Theorem[6]: β_I is a congruence on a Pre A*-algebra A for any ideal I of A.

2.11. Theorem[6]: For any ideals I and J of a Pre A*-algebra A the following hold.

(1) $I \subseteq J \Rightarrow \beta_I \subseteq \beta_J$

(2) β_I I $\beta_J = \beta_{I I J}$

(3) $\beta_I \vee \beta_J = \beta_{I \vee J}$

2.12. Theorem: Let F(L) be the lattice of all ideals a Pre A*-Lattice L. Then $I \rightarrow \beta_I$ is homomorphism of the lattice F(L) into the lattice Con(L) of all congruences on L.

Proof: From 2.11. Theorem (2 and 3) it follows that $I \rightarrow \beta_I$ is lattice homomorphism of F(L) into the lattice Con(L).

2.13. Lemma: Let L be a distributive Pre A*-Lattice. Then $M_a = \{ a \vee x \;/\; x \in L\}$ is a sub algebra of L and it is a distributive lattice.

Proof: Let $a \vee x$, $a \vee y \in M_a$

Then $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \in M_a$ Hence M_a is closed under $\wedge$

Also $a \vee (x \vee y) = (a \vee x) \vee (a \vee y) \in M_a$ Hence M_a is closed under $\vee$

Therefore M_a is a sub algebra of L. Since L is a distributive Pre A*-Lattice we have M_a is a distributive lattice.

2.14. Theorem: Let L be a distributive Pre A*-Lattice. Then the map $g_a : L \to M_a$ defined by $g_a(x) = a \vee x$ is a homomorphism and $L / \beta_a \cong M_a$.

Proof: Let $x, y \in L$. Then

$g_a(x \vee y) = a \vee (x \vee y) = (a \vee x) \vee (a \vee y) = g_a(x) \vee g_a(y)$ and

$g_a(x \wedge y) = a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) = g_a(x) \wedge g_a(y)$

Therefore g_a is homomorphism.

For $a \vee x \in M_a$, $g_a(x) = a \vee x$

Hence g_a is onto.

Now $\ker g_a = \{(x, y) / g_a(x) = g_a(y)\} = \{(x, y) / a \vee x = a \vee y\} = \beta_a$

By fundamental theorem of homomorphism $L / \ker g_a \cong M_a$, which imply $L / \beta_a \cong M_a$.

2.15. Lemma: Let L be a distributive Pre A*-Lattice. Then $L_a = \{ a \wedge x / x \in L\}$ is a sub algebra of L and it is a distributive lattice.

2.16. Theorem: Let L is a distributive Pre A*-Lattice and $a \in L$.Then the mapping $f \to L_a \times M_a$ defined by $f(x) = (a \wedge x, a \vee x)$ for all $x \in L$ is an isomorphism.

Proof: Let $x, y \in L$.

$$
\begin{aligned}
f(x \wedge y) &= (a \wedge (x \wedge y), a \vee (x \wedge y)) \\
&= ((a \wedge x) \wedge (a \wedge y), (a \vee x) \wedge (a \vee y)) = (a \wedge x, a \vee x) \wedge (a \wedge y, a \vee y) \\
&= f(x) \wedge f(y) \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
f(x \vee y) &= (a \wedge (x \vee y), a \vee (x \vee y)) \\
&= ((a \wedge x) \vee (a \wedge y), (a \vee x) \vee (a \vee y)) = (a \wedge x, a \vee x) \vee (a \wedge y, a \vee y) \\
&= f(x) \vee f(y)
\end{aligned}
$$

Hence f is homomorphism

For $a \wedge x \in L_a$ and $a \vee x \in M_a$ where $x \in L$ such that $f(x) = (a \wedge x, a \vee x) \in L_a \times M_a$

Hence f is onto

Let $f(x) = f(y) \Rightarrow (a \wedge x, a \vee x) = (a \wedge y, a \vee y) \Rightarrow a \wedge x = a \wedge y$ and $a \vee x = a \vee y$
$\Rightarrow x = y$

Hence f is one – one.Therefore f is isomorphism.

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